THE SECOND COHOMOLOGY OF SMALL IRREDUCIBLE MODULES FOR SIMPLE ALGEBRAIC GROUPS

GEORGE J. MCNINCH

Abstract. Let G be a simple, simply connected and connected algebraic group over an algebraically closed field of characteristic $p > 0$, and let V be a rational G-module such that $\dim V \leq p$. According to a result of Jantzen, V is completely reducible, and $H^1(G, V) = 0$. In this paper we show that $H^2(G, V) = 0$ unless some composition factor of V is a non-trivial Frobenius twist of the adjoint representation of G.

1. Introduction

Let G be a quasisimple, connected, and simply connected algebraic group over the algebraically closed field k of characteristic $p > 0$. By a G-module V, we always understand a rational G-module (one given by a morphism of algebraic groups $G \to \mathrm{GL}(V)$). In this paper, we study the cohomology of a G-module V such that $\dim V \leq p$. By results of Jantzen [Jan96] one knows that V is semisimple and that $H^1(G, V) = 0$.

Recall that the Lie algebra $\mathfrak{g}$ of G is a G-module via the adjoint action. Our main result is:

Theorem A. *Let V be a G-module with $\dim V \leq p$. Then $H^2(G, V) \neq 0$ if and only if V has a composition factor isomorphic with a* Frobenius twist $\mathfrak{g}^{[d]}$ *of $\mathfrak{g}$ for some $d \geq 1$.*

Differentiating the representation of G on V gives a representation for the Lie algebra $\mathfrak{g}$ on V. Assume that $V^{\mathfrak{g}} = 0$. Then the theorem says that $H^2(G, V) = 0$. For V of this sort, the vanishing of H^2 is a consequence of the linkage principle for G together with results in section 2 which give estimates for the dimensions of Weyl modules whose high weights are simultaneously in the low alcove and in the orbit $W_p \bullet 0$. In fact, the same argument shows that $H^i(G, V)$ is 0 for all $i \geq 1$; see Proposition 5.2. It was pointed out to me that an earlier version of this manuscript contained an overly complicated proof of this observation.

The crucial case for Theorem A is when V is simple, non-trivial and $V^{\mathfrak{g}} = V$. There is a unique $d \geq 1$ such that the "Frobenius untwist" $V^{[-d]}$ is a G-module on which $\mathfrak{g}$ acts non-trivially. We have already seen that $H^i(G, V^{[-d]}) = 0$ for $i = 1, 2$, so Theorem A follows from the following two results (see 5.4). [We denote by h the Coxeter number of the group G.]

Theorem B. *Suppose that $p \geq h$ and that W is a G-module for which $H^i(G, W) = 0$ for $i = 1, 2$. Then $H^2(G, W^{[d]}) \simeq \mathrm{Hom}_G(\mathfrak{g}, W)$ for all $d \geq 1$.*

Date: February 1, 2008.

2000 *Mathematics Subject Classification.* 20G05.

This work was supported by a grant from the National Science Foundation.

Theorem C. *If $p > h$, $\dim H^2(G, \mathfrak{g}^{[d]}) = 1$ for all $d \geq 1$. For any p, there is a $d_0 \geq 1$ so that $H^2(G, \mathfrak{g}^{[d]}) \neq 0$ for all $d \geq d_0$.*

Theorem B is proved in 5.3; it immediately implies the first assertion of Theorem C (see 5.5). We give a proof the second assertion of Theorem C in section 5.6.

We end the paper by applying the results of section 2 to calculations of cohomology groups $H^i(G_1, L)$, where G_1 is the Frobenius kernel, and L is a simple G_1 module with $\dim L \leq p$; see Proposition 6.

We conclude this introduction by remarking that the result of Jantzen [Jan96] cited above is one of several recent results studying the semisimplicity of low dimensional representations of groups in characteristic p. See [Ser94], [McN98], [McN99],[Gur99], and [McN00] for related work.

The author would like to acknowledge the hospitality of Bob Guralnick and the University of Southern California during a visit in June 1999; in particular, questions of Guralnick encouraged the author to consider the problems addressed in this paper, and several conversations inspired some useful ideas.

2. Root systems

2.1. We denote by R an indecomposable root system in its weight lattice X with simple roots $S \subset R^+$. For each $\alpha \in S$, there is a fundamental dominant weight $\varpi_\alpha \in X$; the ϖ_α form a $\mathbb{Z}$ basis of X.

We write α_0 for the dominant short root, and $\tilde{\alpha}$ for the dominant long root in R (these coincide in case there is only one root length).

The Coxeter number of R is given by

$$h - 1 = \sup_{\alpha \in R^+} \{\langle \rho, \alpha^\vee \rangle\} = \langle \rho, \alpha_0^\vee \rangle.$$

For $m \in \mathbb{Z}$ and $\alpha \in R$, let $s_{\alpha,m}$ denote the affine reflection of $X_\mathbb{R} = X \otimes_\mathbb{Z} \mathbb{R}$ in the hyperplane $H_{\alpha,m} = \{x \in X_\mathbb{R} : \langle x, \alpha^\vee \rangle = m\}$.

Let $l > h$ be an integer. The affine Weyl group W_l is the group of affine transformations of $X_\mathbb{R}$ generated by all $s_{\alpha,ln}$ for $n \in \mathbb{Z}$. According to [Bou72, ch. VI, §2.1, Prop. 1] W_l is isomorphic to the semidirect product of W (the finite Weyl group) with $l\mathbb{Z}R$. The normalizer of W_l in the full affine transformation group of $X_\mathbb{R}$ contains all translations by lX, hence W_l is a normal subgroup of $\widehat{W}_l$, the semidirect product of W and lX. Moreover, $\widehat{W}_l/W_l \simeq lX/l\mathbb{Z}R \simeq X/\mathbb{Z}R$ is the fundamental group of R, which we will denote by π.

Let $\rho = \dfrac{1}{2}\sum_{\alpha \in S} \alpha$. We always consider the dot action of $\widehat{W}_l$ (also of W and W_l) on X: for $w \in \widehat{W}_l$ and $\lambda \in X$, this is given by $w \bullet \lambda = w(\lambda + \rho) - \rho$.

The subset C_l of $X_\mathbb{R}$ given by

$$C_l = \{\lambda \in X_\mathbb{R} \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < l \quad \text{for each } \alpha \in R^+\}.$$

is a fundamental domain for the dot action of W_l on X; its conjugates under W_l are known as alcoves, and C_l is the lowest alcove. Since $\widehat{W}_l$ normalizes W_l, [Bou72, ch. VI, §2.1] shows that $\widehat{W}_l$ permutes the alcoves.

Let Ω be the stabilizer in $\widehat{W}_l$ of C. Since W_l permutes the alcoves simply transitively, one deduces that $\widehat{W}_l$ is the semidirect product of Ω and W_l. Thus $\Omega \simeq \widehat{W}_l/W_l \simeq \pi$.

Since $l > h$, the intersection $C_l \cap X^+$ is non-empty. [Note that if $l \le h$ had been allowed, we would have $C_l \cap X^+ = \{0\}$ in case $l = h$, and $C_l \cap X^+ = \emptyset$ if $l < h$.] It is then clear that $\widehat{W}_l \bullet 0 \cap C_l = \{\omega \bullet 0 \mid \omega \in \Omega\}$.

2.2. Let I index the simple roots $S = \{\alpha_i\}$, write $\alpha_0^\vee = \sum_{i \in I} n_i \alpha_i^\vee$, and put $J = \{i \in I \mid n_i = 1\}$. A dominant weight $0 \ne \varpi \in X$ is *minuscule* if whenever $\lambda \le \varpi$ and λ is a dominant weight, then $\varpi = \lambda$. According to [Bou72, Ch. VI, exerc. 23,24], ϖ is minuscule just in case $\varpi = \varpi_i$ for some $i \in J$.

For $i \in I \cup \{0\}$, let $S_i = S \setminus \{\alpha_i\}$ (so $S_0 = S$). Write $R_i \subset R$ for the root subsystem determined by S_i, and W_i for the parabolic subgroup of W associated with R_i. Let $w_i \in W_i$ be the unique element which makes all positive roots in R_i negative.

For $x \in X$, let $t(x)$ denote the affine translation by x; for $i \in J$, let $\gamma_i = t(l\varpi_i)w_0 w_i \in \widehat{W}_l$. Note that γ_i represents $\varpi_i \in X/\mathbb{Z}R \simeq lX/l\mathbb{Z}R \simeq \widehat{W}_l/W_l$.

Applying [Bou72, ch. VI, §2.2 Prop. 6 and Cor.] one obtains:

Proposition. (a) *Each non-0 coset of $\mathbb{Z}R$ in X is uniquely represented by a minuscule weight. In particular, $|\pi| = |J| + 1$.*

(c) *The non-identity elements of Ω are precisely the γ_i for $i \in J$. We have*

$$\widehat{W}_l \bullet 0 \cap C_l = \{0\} \cup \{\gamma_i \bullet 0 = (l-h)\varpi_i \mid i \in J\}$$

2.3. For a dominant weight λ, let

$$d(\lambda) = \prod_{\alpha > 0} \frac{\langle \lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} \tag{1}$$

be the value of Weyl's degree formula at λ.

Proposition. *Let $\lambda = (l-h)\varpi_i$ for some $i \in J$.*

(a) *$d(\lambda) \ge \binom{l-1}{l-h}$, with equality if and only if $h - 1 = \ell(w_0 w_i)$.*

(b) *If $l - h \ge 2$ and $h \ge 3$, then $d(\lambda) > l$.*

Proof. For $1 \le k \le h-1$, let $e(k)$ be the number of $\alpha \in R^+ \setminus R_i^+$ with $\langle \rho, \alpha^\vee \rangle = k$. The argument in the remark on p. 520-521 of [Ser94] (following Prop. 6) shows that $e(k) \ge 1$ for each $1 \le k \le h-1$. Thus, we have

$$d(\lambda) = \prod_{k=1}^{h-1} \left(\frac{l-h+k}{k} \right)^{e(k)} \ge \prod_{k=1}^{h-1} \frac{l-h+k}{k} = \binom{l-1}{l-h}.$$

If $\ell(w_0 w_i) = |R^+| - |R_i^+| = h-1$, then $e(k) = 1$ for each $1 \le k \le h-1$ and equality holds. This proves (a).

For (b), note that under the given hypothesis we have $l \ge 5$. Since $\binom{l-1}{l-h} \ge \binom{l-1}{2} > l$ for all such l, (b) follows immediately. ∎

Remark. Using the table in the proof of Proposition 2.4 below, it is straightforward to verify that equality holds in (a) if and only if either $R = A_r$ and $i \in \{1, r\}$ or $R = C_r$ and $i = 1$. (Since $B_2 = C_2$, the latter case includes B_2 and $i = 2$.)

2.4. In the following, let me emphasize the standing assumption $l > h$.

Proposition. *If $0 \neq \lambda \in \widehat{W}_l \bullet 0 \cap C$ and $d(\lambda) < l$ then $d(\lambda) = \ell - 1$ and (R, λ) is listed in the following table. If the rank of R is ≥ 2, then $l = h + 1$.*

R	l	λ
A_1	any	$(l-2)\varpi_1$
A_{l-2}		ϖ_1, ϖ_{l-2}
B_2	$l = 5$	ϖ_2
$C_{(l-1)/2}$	l odd	ϖ_1

Proof. The rank 1 situation leads to the item listed in the table. When the rank is at least 2, one applies Proposition 2.3 to obtain $l = h + 1$, whence $\lambda = \varpi_i$ for some $i \in J$; i.e. λ is minuscule.

We handle the minuscule cases by classification. For each indecomposable root system R for which $J \neq \emptyset$, we list in the following table the Coxeter number, the set J, and the value $d(\varpi_i)$ for each $i \in J$. The simple roots are indexed as in the tables in [Bou72, Planche I-X]; the data recorded here, with the exception of the values $d(\varpi_i)$, may be verified by inspecting those tables as well. The values $d(\varpi_i)$ are well known (and can anyway be computed from the formula, or by representation theoretic arguments).

Type of R	h	J	$d(\varpi_i),\ i \in J$
A_r	$r+1$	$\{1, 2, \ldots, r\}$	$\binom{r+1}{i}$
$B_r, r \geq 2$	$2r$	$\{r\}$	2^r
$C_r, r \geq 2$	$2r$	$\{1\}$	$2r$
$D_r, r \geq 4$	$2r-2$	$\{1, r-1, r\}$	$2r, 2^{r-1}, 2^{r-1}$ respectively
E_6	12	$\{1, 6\}$	$27, 27$
E_7	18	$\{7\}$	56

From this table, one can list all pairs (R, λ) for which R has Coxeter number $l - 1$ and λ is minuscule. It is a simple matter to see that $d(\lambda) < l$ only when (R, λ) is as claimed. ☐

3. The algebraic groups

3.1. Let k be an algebraically closed field of characteristic $p > 0$, and let G be a connected, simply connected semisimple algebraic k-group. The non-0 weights of a maximal torus $T \leq G$ on $\mathfrak{g} = \mathrm{Lie}(G)$ form an indecomposable root system R of rank $r = \dim T$ in the character group $X = X^*(T)$. Since G is simply connected, X identifies with the full weight lattice of R as in section 2. We fix a choice of simple roots S and positive roots R^+. The dominant weights are denoted X^+. The group G is assumed to be *quasisimple*; i.e. the root system R is indecomposable.

3.2. For each dominant weight $\lambda \in X^+$, the space of global sections of the corresponding line bundle on the flag variety affords an indecomposable rational G-module $H^0(\lambda)$ with simple socle. The modules $L(\lambda) = \operatorname{soc} H^0(\lambda)$ comprise all of the simple rational modules for G (and are pairwise non-isomorphic).

The character of each $H^0(\lambda)$ is the same as in characteristic 0; hence in particular $\dim_k H^0(\lambda)$ is given by the Weyl degree formula, whose value at λ we denote $d(\lambda)$ as in 2.3.

3.3. Any dominant λ may be written as a finite sum $\sum_{i\geq 0} p^i\lambda_i$ with each λ_i a *restricted* weight. Recall that a dominant weight μ if $\langle \mu, \alpha^\vee\rangle < p$ for all simple roots α. Steinberg's tensor product theorem says:

$$L(\lambda) \simeq L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes L(\lambda_2)^{[2]} \otimes \cdots$$

where for a G-module V, $V^{[m]}$ standards for the m-th Frobenius twist of V.

For $d \geq 1$, let G_d be the d-th Frobenius kernel of G. Let V be a rational G-module and $m \geq 1$. If there is a rational G module W with $W^{[m]} \simeq V$, we regard W as the Frobenius *untwist* $W = V^{[-m]}$ of V. Now regard V as a module for G_d. Since G_d is a normal subgroup scheme, G acts on V^{G_d}; since G_d acts trivially on this G-module, there is an untwisted rational G-module $(V^{G_d})^{[-d]}$. It follows that there is an untwist $H^i(G_d, V)^{[-d]}$ for all $i \geq 0$.

Consider now two G-modules V_1 and V_2, and form $W = V_1 \otimes V_2^{[d]}$. The Frobenius kernel G_d acts trivially on $V_2^{[d]}$, so that

$$H^i(G_d, W)^{[-d]} \simeq H^i(G_d, V_1)^{[-d]} \otimes V_2 \tag{1}$$

as G-modules for every $i \geq 0$.

3.4. Let $W_p \leq \widehat{W}_p$ be as in section 2 (for $l = p$), let $C = C_p \cap X^+$ denote the dominant weights in the lowest alcove, and let $\bar{C} = \bar{C}_p \cap X^+$ ($\bar{C}_p$ is the closure in $X_{\mathbb{R}}$).

Proposition. *Let $\lambda \in X^+$.*

(a) *If $H^i(G, L(\lambda)) \neq 0$ for some $i \geq 0$. then $\lambda \in W_p \bullet 0$.*
(b) *If $H^i(G_1, L(\lambda)) \neq 0$ for some $i \geq 0$, then $\lambda \in \widehat{W}_p \bullet 0$.*
(c) *$H^i(G, H^0(\lambda)) = 0$ for all $i > 0$.*
(d) *If $\lambda \in \bar{C}$, then $L(\lambda) = H^0(\lambda)$; in particular, $\dim L(\lambda) = d(\lambda)$.*

Proof. (a) follows from the *linkage principle* for G [Jan87, Cor. II.6.17], and (b) from the linkage principle for G_1 [Jan87, Lemma II.9.16]. (c) follows from [Jan87, II.4.12]. (d) follows from [Jan87, II.6.13,II.5.10]. ☐

4. The Lie algebra and the cohomology of G_1

We want to describe explicitly the cohomology $H^*(G_1, k)$ in degree ≤ 2. For this, we need some information on the Lie algebra $\mathfrak{g}$.

4.1. Recall that the prime p is *bad*[=not good] for the indecomposable root system R if one of the following holds: $p = 2$ and R is not of type A_r; $p = 3$ and R is of type G_2,F_4, or E_r; $p = 5$ and R is of type E_8.

The prime p is *very good* if it is not bad, and in case $R = A_r$, if also p does not divide $r + 1$.

Application of the summary in [Hum95, 0.13] yields:

Lemma A. *Assume that p is* very good. *Then $\mathfrak{g}$ is a simple Lie algebra. The adjoint G-module is simple, self-dual, and isomorphic with $L(\tilde{\alpha})$ where $\tilde{\alpha}$ is the dominant long root.*

Notice that if $p > h$, then p is very good.

Lemma B. *Assume that $p \geq h$. If W is any G-module, then* $\mathrm{Hom}_G(\mathfrak{g}, W^{[d]}) = 0$ *for $d \geq 1$.*

Proof. When $p > h$ this follows since $\mathfrak{g}$ is a simple $\mathfrak{g}$-module with restricted highest weight. When $p = h$, we have $R = A_{p-1}$. Since G is simply connected, we have $\mathfrak{g} = \mathfrak{sl}_p$. Thus $\mathfrak{g}$ is an indecomposable G-module with unique simple quotient $L(\tilde{\alpha})$, and the lemma follows. □

4.2. Let B be a Borel subgroup of G, and let $\mathfrak{u}$ be the nilradical of $\mathrm{Lie}(B)$. Regarding $\mathfrak{u}^*$ as a B-module, we get a vector bundle on G/B which we also write as $\mathfrak{u}^*$. According to [AJ84, 3.8], the formal character of the G-module $H^0(G/B, \mathfrak{u}^*)$ is $\chi(\tilde{\alpha}) = \mathrm{ch}(\mathfrak{g}^*)$.

Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone. There is by [AJ84, 3.9] an injective homomorphism of graded algebras $k[\mathcal{N}] \to H^0(G/B, S\mathfrak{u}^*)$

Lemma. *For simply connected, quasisimple algebraic groups G, $\mathfrak{g}^* \simeq k[\mathcal{N}]_1 \simeq H^0(G/B, \mathfrak{u}^*)$.*

Proof. Let $I(\mathcal{N}) \lhd k[\mathfrak{g}] = S\mathfrak{g}^*$ be the (homogeneous) defining ideal of the variety $\mathcal{N}$. We need to show that $I(\mathcal{N})_1 = 0$. If not, then $\mathcal{N} \subset V \subset \mathfrak{g}$ for some proper G-submodule V. A look at the summary in [Hum95, 0.13] shows that, since G is simply connected, the only G-submodules of $\mathfrak{g}$ have dimension 0 or 1. On the other hand, by [Hum95, Theorem 6.19], the variety $\mathcal{N}$ has codimension $\mathrm{rank}(G)$ in $\mathfrak{g}$ and so clearly can't be contained in a 1 dimensional linear subspace! □

Remarks.
1. Here is a fancier result which implies the lemma if we assume that the prime p is good for G. Since G is simply connected and p is good, the Springer resolution
$$\varphi : \tilde{\mathcal{N}} = G \times^B \mathfrak{u} \to \mathcal{N}$$
given by $(g, X) \mapsto \mathrm{Ad}(g)(X)$ is a *desingularization*, hence in particular a birational map; see [Hum95, Theorem 6.3 and Theorem 6.20]. Again since G is simply connected and p is good, the variety $\mathcal{N}$ is normal ([Hum95, Theorem 4.24]). Standard arguments then yield an isomorphism of graded algebras $k[\mathcal{N}] \xrightarrow[\simeq]{\varphi^*} \Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}})$. Finally, the projection $\tilde{\mathcal{N}} \to G/B$ is an affine morphism, so that $\Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}) = H^0(G/B, S\mathfrak{u}^*)$ as a graded algebra.
2. On the other hand, if $G = PGL_r$, and $p|r$, one can find a linear form on $\mathfrak{g}$ that vanishes on $\mathcal{N}$, hence there can be no isomorphism $k[\mathcal{N}]_1 \to H^0(G/B, \mathfrak{u}^*)$ (compare formal characters). So the lemma can fail when G is not simply connected. [Note that φ is not birational in this example. One can show that there is a G_{sc}-isomorphism $\psi : \tilde{\mathcal{N}}_{sc} \to \tilde{\mathcal{N}}$ (using some obvious notations). We get therefore a commuting diagram:
$$\begin{tikzcd} \tilde{\mathcal{N}} \arrow[r, "\varphi_{sc}\circ\psi^{-1}"] \arrow[dr, "\varphi"'] & \mathcal{N}_{sc} \arrow[d, "\gamma"] \\ & \mathcal{N} \end{tikzcd}$$
The map $\varphi_{sc} \circ \psi^{-1}$ is birational. Since $\gamma^* k(\mathcal{N}) \subset k(\mathcal{N}_{sc})$ is a proper purely inseparable extension, so too is $\varphi^* k(\mathcal{N}) \subset k(\tilde{\mathcal{N}})$.]

Proposition.
1. *If $p \neq 2$ or if R is not of type C_r, then $H^1(G_1, k) = 0$.*
2. *Assume that $p \geq h$. Then $H^2(G_1, k)^{[-1]} \simeq \mathfrak{g}^*$ as G-modules.*

Proof. For (1) see [Jan87, Lemma II.12.1]. For (2), first suppose that $p > h$. By [AJ84, 3.7,3.9], there is a G-equivariant isomorphism of graded rings $k[\mathcal{N}]' \simeq H^*(G_1, k)^{[-1]}$ where $k[\mathcal{N}]'$ is again the graded coordinate ring of $\mathcal{N}$, but with the linear functions on $\mathfrak{g}$ given degree 2. The claim now follows from the lemma.

When $p = h$, apply [AJ84, Cor. 6.3] to see that $H^2(G_1, k)^{[-1]} \simeq H^0(G/B, \mathfrak{u}^*)$; the claim follows again from the lemma in this case. □

5. Low dimensional modules for G

5.1. We recall first some facts about low dimensional modules established in [Jan96] and [Ser94].

Proposition. *Let L be a simple non-trivial restricted G module with highest weight λ. Suppose that* $\dim L \leq p$.

(a) $\lambda \in \bar{C}$.
(b) $\lambda \in C$ *if and only if* $\dim_k L < p$.
(c) $h \leq p$. *If moreover* $\dim L < p$, *then* $h < p$.
(d) *If R is not of type A and* $\dim L = p$, *then* $h < p$. *If* $p = h$ *and* $\dim L = p$, *then* $R = A_{p-1}$ *and* $\lambda = \varpi_i$ *with* $i \in \{1, p-1\}$.

Proof. (a) follows from [Jan96, Lemma 1.4], and (b) from [Jan96, 1.6], see also [Ser94]. For (c), note first that (a) implies $\dim L = d(\lambda)$ by Proposition 3.4(d). If $\lambda \in \bar{C} \setminus C$, then (a) and (b) imply that $\dim L = p$, whence $p = h$ follows from Weyl's degree formula. (c) now follows since C is empty if $p < h$ and $C = \{0\}$ if $p = h$.

In [Jan96, 1.6], Jantzen made a list of all simple restricted modules for G with dimension p. Inspecting that list yields (d). □

5.2. **Vanishing results when $\mathfrak{g}$ acts non-trivially.** Let L be a simple module for G.

Proposition. *If G_1 (equivalently: $\mathfrak{g}$) acts non-trivially on L and* $\dim L \leq p$, *then* $H^i(G, L) = 0$ *for all* $i \geq 0$.

Proof. Write the highest weight of L as $\lambda = \mu_1 + p\mu_2$ with μ_1 restricted. Since $L^{\mathfrak{g}} = 0$, we have $\mu_1 \neq 0$. Since $p \geq \dim L \geq \dim L(\mu_1)$, Proposition 5.1 implies that $\mu_1 \in \bar{C}$ and that $h \leq p$. We have in particular that $L(\mu_1) = H^0(\mu_1)$, hence the proposition will follow from Proposition 3.4 if we show that μ_2 is 0.

If $\dim L = p$, Steinberg's tensor product theorem gives $\mu_2 = 0$. If $\dim L < p$ then 5.1 shows that $p < h$ and $\mu_1 \in C$. If $H^i(G, L) \neq 0$ for some i, then $\lambda \in W_p \bullet 0$ by the linkage principle, whence $\mu_1 \in W \bullet 0 + pX = \widehat{W_p} \bullet 0$. Now Proposition 2.4 applies; it shows that $\dim L(\mu_1) = p - 1$ whence we have $\mu_2 = 0$ by another application of Steinberg's theorem. □

5.3. **Second cohomology.** Here we prove our main tool for describing second cohomology; first we require the following:

Lemma. *Let $E_2^{p,q} \implies H^{p+q}$ be a convergent, first quadrant spectral sequence.*

1. *If $E_2^{0,1} = E_2^{1,1} = E_2^{0,2} = 0$, then $H^2 \simeq E_2^{2,0}$*
2. *If $E_2^{1,0} = E_2^{1,1} = E_2^{2,0} = 0$, then $H^2 \simeq E_2^{0,2}$.*

Proof. We verify (1), the argument for (2) is the same. We must show that $E_\infty^{2,0} \simeq E_2^{2,0}$; first note that $E_3^{2,0}$ is the cohomology of the sequence

$$E_2^{0,1} \to E_2^{2,0} \to E_2^{4,-1}$$

from which we get $E_3^{2,0} \simeq E_2^{2,0}$. For any first quadrant spectral sequence one has (by similar reasoning) that $E_a^{2,0} \simeq E_{a+1}^{2,0}$ for $a > 2$, so we get the desired isomorphism. $\square$

Theorem. *Suppose that $p \geq h$. Let V be a G-module for which $H^i(G, V) = 0$ for $i = 1, 2$, and let $d \geq 1$.*

1. *$H^1(G, V^{[d]}) = 0$, and*
2. *$H^2(G, V^{[d]}) \simeq \mathrm{Hom}_G(\mathfrak{g}, V)$.*

Proof. The Frobenius kernel G_1 is a normal subgroup of G; thus there is a Lyndon-Hochschild-Serre spectral sequence computing $H^i(G, V^{[d]})$ which in view of 3.3 (1) has the form

$$E_2^{s,t} = H^s(G, H^t(G_1, V^{[d]})^{[-1]}) = H^s(G, H^t(G_1, k)^{[-1]} \otimes V^{[d-1]})$$

If $t = 1$, $E_2^{s,t} = 0$ by Lemma 4.2(1).

There is an exact sequence of the form [Jan87, I.4.1(4)]

$$0 \to E_2^{1,0} \to H^1(G, V^{[d]}) \to E_2^{0,1} = 0.$$

Thus $H^1(G, V^{[d]}) \simeq E_2^{1,0} \simeq H^1(G, V^{[d-1]})$. We get now (1) by induction on d.

Lemma 4.2(2) shows now that $H^2(G_1, k) \simeq \mathfrak{g}^*$. Thus, the only possible non-0 E_2 terms of total degree 2 are

$$\begin{aligned} E_2^{0,2} =& H^0(G, \mathfrak{g}^* \otimes V^{[d-1]}) = \mathrm{Hom}_G(\mathfrak{g}, V^{[d-1]}) \\ E_2^{2,0} =& H^2(G, V^{[d-1]}). \end{aligned}$$

For $d > 1$, we apply 4.1 Lemma B to see that $E_2^{0,2} = 0$ whence $H^2(G, V^{[d]}) \simeq E_2^{2,0} = H^2(G, V^{[d-1]})$ by part (1) of the lemma; thus (2) will follow provided it holds for $d = 1$. In that case, we have $E_2^{2,0} = 0$ by assumption, and the result just proved in part (1) shows that $E_2^{1,0} = 0$. Thus part (2) of the lemma applies; it shows that $H^2(G, V^{[1]}) \simeq E_2^{0,2} = \mathrm{Hom}_G(\mathfrak{g}, V)$ as desired. $\square$

5.4. **The second cohomology of small modules.** Let $L = L(\lambda)$ be a simple G-module, and suppose that $\dim L \leq p$. Proposition 5.2 showed that the vanishing of cohomology for L is a consequence of the linkage principle when $\lambda \notin pX$. However, if $\lambda \in p\mathbb{Z}R$, λ is linked to 0, so the linkage principle does not yield vanishing. The following result shows that, despite the linkage of λ and 0 in this case, the second cohomology is usually 0.

Theorem. *Let L be a simple G-module with $\dim L \leq p$. If $H^2(G, L) \neq 0$, then $L \simeq \mathfrak{g}^{[d]}$ for some $d \geq 1$.*

Proof. Let L' be such that $L \simeq (L')^{[d]}$ for $d \geq 0$, and such that $\mathfrak{g}$ acts non-trivially on L'. We have by 5.1 that $p \geq h$. Also, we have by Proposition 5.2 that $H^i(G, L') = 0$ for $i \geq 1$. If $d = 0$, we are done. If $d > 1$, then Theorem 5.3 applies, and we get that

$$H^2(G, L) \simeq \mathrm{Hom}_G(\mathfrak{g}, L').$$

We get by Proposition 5.1 that $p > h$ unless $R = A_{p-1}$ and $L' = L(\varpi_i)$ with $i \in \{1, p-1\}$. If $p > h$, then $\mathfrak{g}$ is a simple G-module by Lemma 4.1. So if $\mathrm{Hom}_G(\mathfrak{g}, L') \neq 0$ then $L' \simeq \mathfrak{g}$ whence $L \simeq \mathfrak{g}^{[d]}$ as claimed.

In the remaining case, one must just note that weight considerations yield $\mathrm{Hom}_G(\mathfrak{g}, L(\varpi_i)) = 0$ for $i = 1, p-1$, whence $H^2(G, L) = 0$. □

5.5. **The second cohomology of twists of the adjoint module.** The first assertion of Theorem C of the introduction follows from the following:

Proposition. *Assume that $p > h$. Then $H^1(G, \mathfrak{g}^{[d]}) = 0$ and $H^2(G, \mathfrak{g}^{[d]}) \simeq \mathrm{End}_G(\mathfrak{g})$ has dimension 1 for $d \geq 1$.*

Proof. Since $p > h$, Lemma 4.1 shows that $\mathfrak{g}$ is the simple module with highest weight $\tilde{\alpha}$. It follows that $\mathfrak{g} = H^0(\tilde{\alpha})$, and thus that $H^i(G, \mathfrak{g}) = 0$ for $i \geq 1$ by Proposition 3.4. The proposition now follows from Theorem 5.3. □

Remark. Note that $\dim \mathfrak{g} > h$ (in fact, $\dim \mathfrak{g} = (h+1)r$ where r is the rank of G). So we get also: if $\dim \mathfrak{g} \leq p$, then $\dim H^2(G, \mathfrak{g}^{[d]}) = 1$ for $d \geq 1$.

5.6. **A second proof.** Here we give a second proof of the non-vanishing of H^2 for twists of the adjoint module; the result proved here verifies the remaining assertion of Theorem C of the introduction. We have included the argument since it offers some "explanation" for the non-vanishing.

The group G arises by base change from a split reductive group scheme $\mathbf{G}$ over $\mathbb{Z}$. Let $\mathbb{Z}_p$ be the complete ring of p-adic integers, and let $\mathbb{Q}_p$ be its field of quotients. For any finite field extension F of $\mathbb{Q}_p$, let $\mathfrak{o}$ denote the integers in F. The residue field $\mathfrak{o}/\mathfrak{m}$ may be identified with the extension $\mathbb{F}_q$ of $\mathbb{F}_p$.

Let K denote the group of points $\mathbf{G}(\mathfrak{o})$ regarded as a subgroup of $\mathbf{G}(F)$. Since $\mathbf{G}$ is smooth, the reduction homomorphism $K \to \mathbf{G}(\mathbb{F}_q)$ is surjective (see [Tit79, 3.4.4]).

For $n \geq 1$, let $K_n \subset K$ be the kernel of the map $K \to \mathbf{G}(\mathfrak{o}/\mathfrak{m}^n)$. Note that $K/K_1 = \mathbf{G}(\mathbb{F}_q)$ acts by conjugation on each quotient K_n/K_{n+1}.

Proposition. (a) *There is for each $m \geq 1$ a canonical isomorphism $K_m/K_{m+1} \simeq \mathfrak{g}_{\mathbb{F}_q}$ as representations for $\mathbf{G}(\mathbb{F}_q)$, where $\mathfrak{g}_{\mathbb{F}_q}$ is the Lie algebra of $\mathbf{G}_{\mathbb{F}_q}$.*

(b) *If $H^2(\mathbf{G}(\mathbb{F}_q), \mathfrak{g}_{\mathbb{F}_q}) = 0$, the exact sequence of groups*

$$1 \to K_1 \to K \to \mathbf{G}(\mathbb{F}_q) \to 1$$

splits.

(c) *There is a p-power q_0, depending only on the root system R of G, such that $H^2(\mathbf{G}(\mathbb{F}_q), \mathfrak{g}_{\mathbb{F}_q}) \neq 0$ whenever $q \geq q_0$.*

(d) *There is an integer $a_0 \geq 1$ such that $H^2(G, \mathfrak{g}^{[a]}) \neq 0$ whenever $a \geq a_0$.*

Proof. (a) Follows from [DG70, II.§4.3]. (b) Since K_1 is a pro-p group [PR94, Lemma 3.8], this follows from [Ser67, Lemma 3].

(c) Choose a $\mathbb{Q}_p$ vectorspace V and a non-trivial faithful $\mathbb{Q}_p$-rational representation $\mathbf{G}_{\mathbb{Q}_p} \to \mathrm{GL}(V)$. For each extension F of $\mathbb{Q}_p$ with integers $\mathfrak{o}$, the group $K = \mathbf{G}(\mathfrak{o})$ is a subgroup of (the group of F-points of) $\mathrm{GL}(V_F)$. If $H^2(\mathbf{G}(\mathbb{F}_q), \mathfrak{g}_{\mathbb{F}_q}) = 0$, the sequence in (b) is split and V_F is a non-trivial $F[\mathbf{G}(\mathbb{F}_q)]$-module.

Since F has characteristic 0, it is well known that the minimal dimension of a non-trivial $F[\mathbf{G}(\mathbb{F}_q)]$ module is bounded below by the value $f(q)$ of a polynomial $f \in \mathbb{Q}[x]$, depending only on G, for which $f(q) \to \infty$ as $q \to \infty$. We may choose q_0 such that $f(q) > \dim_{\mathbb{Q}_p} V$ for each $q > q_0$, and (c) follows at once.

(d) now follows from (c) and [CPSvdK77, Cor. 6.9]. □

6. Small simple modules for G_1

Combining results of [KLT99] with the results recorded in 2.4, we obtain some explicit results on G_1 cohomology of low dimensional simple modules:

Proposition. *Let L be a non-trivial simple G_1 module with $\dim \le p$. Assume for some $i \ge 0$ that $H^i(G_1, L) \ne 0$. Then $\dim L = p-1$. Moreover, there is a quadruple $(R, \lambda, i(0), V)$ in the following table for which R is the root system of G, λ the high weight of L, $i \ge i(0)$ and $H^{i(0)}(G_1, L)^{[-1]} \simeq V$ as G-modules.*

R		λ	$i(0)$	$H^{i(0)}(G_1, L)^{[-1]}$
A_1		$(p-2)\varpi_1$	1	$L(\varpi_1)$
A_{p-2}		ϖ_1, ϖ_{p-2}	$p-2$	$L(\lambda)$
$C_{(p-1)/2}$	p *odd*	ϖ_1	$p-2$	$L(\lambda)$

Proof. By [Jan87, Prop. II.3.14], $L = \mathrm{res}^G_{G_1} L(\lambda)$ for some restricted dominant weight $0 \ne \lambda$. Thus $L(\lambda)$ is a restricted, simple G module with dimension $\le p$. It follows from Proposition 5.1 that $h \le p$, that $\lambda \in \bar{C}$, and that $L = H^0(\lambda)$ as modules for G.

Suppose that $H^i(G_1, L) \ne 0$ for some i. By the linkage principle for G_1 (Proposition 3.4(b)), we must have $\lambda \in \widehat{W_p} \bullet 0$, hence $\lambda \in C$. This implies that $h < p$. Proposition 2.2 shows that $\lambda = (p-h)\varpi_i = w_0 w_i \bullet 0 + p\varpi_i$ for some $i \in J$, and Proposition 2.3 yields $\dim L = p-1$ and lists the possible pairs (R, λ).

For $h < p$, Kumar, Lauritzen and Thomsen [KLT99, Theorem 8] have extended a result of Andersen and Jantzen [AJ84, 3.7]; this result implies in particular that the minimal degree for which $H^*(G_1, L)$ is non-0 is $\ell(w_0 w_i)$, and that

$$H^{\ell(w_0 w_i)}(G_1, L)^{[-1]} \simeq H^0(\varpi_i).$$

It is straightforward to compute for each pair (R, λ) the length $\ell(w_0 w_i)$; one gets in this way the result. □

Remark. The Theorem implies the fact (used by Jantzen in the proof of [Jan96, Lemma 1.7]) that $H^1(G_1, L) = 0$ for all simple G_1 modules L with $\dim L \le p$. The argument used by Jantzen there relied on the calculations of H^1 carried out in [Jan91].

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E-mail address: `McNinch.1@nd.edu`

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556 USA